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ON THE DIOPHANTINE EQUATION $P^x + Q^y = Z^2$

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ABSTRACT. *A Diophantine equation is a polynomial equation involving two or more variables for which integral solutions are sought. An exponential Diophantine equation includes additional variables that appear as exponents. This paper focuses on determining integral solutions to the Diophantine equation px + qy = z², given that x + y = 5, where p and q are twin primes, cousin primes, sexy primes, or any positive integers. By analyzing the solution patterns for each scenario, we aim to develop theorems and lemmas. The findings in this paper demonstrate that, for all cases where x + y = 5, the Diophantine equation does not have any non-trivial solutions when p and q are twin primes, cousin primes, or sexy primes, but it does have infinitely many solutions for any positive integers.*

INTRODUCTION

A Diophantine equation is a mathematical expression that involves two or more variables, where only integer solutions are desired. Over the years, extensive research has been conducted in this area, including studies by Cohn (1993), Demirci (2017), and Bennet (2017), as well as investigations into various other types of equations. Burshtein (2017) analyzed solutions to the Diophantine equation $p^x + q^y = z^4$ and demonstrated that for all primes $p \geq 2$ and $y = 1$, the equation has infinitely many solutions for every $x \geq 1$. Specifically, when x is even and q is prime, there exists a unique solution, $(p, q, x, y, z) = (2,17,6,1,3)$. In contrast, when x is even and q is composite, there are multiple solutions. For odd values of x, infinitely many solutions exist regardless of whether q is prime or composite.

Burshtein (2018a) used prime numbers to solve a new equation in the form of $p^x + (p+4)^y = z^2$, focusing on cases where $(p, p+4)$ are cousin primes and $x + y < 5$. He evaluated six scenarios where the sum of x and y is less than 5 and found a solution for $x = 2$ and $y = 1$, specifically $(p, x, y, z) = (3,2,1,4)$. Burshtein (2018b) also addressed the same problem for $p^x + q^y = z^2$ with primes for $k = 3$, and the pairs $(p, p+6)$, termed sexy primes, finding seven solutions among the first 10,000 primes p, where $x = y = 1$. Additionally, there was exactly one solution for $x = 2$ and $y = 1$.

Burshtein (2019) further examined two equations: $5^x + 103^y = z^2$ and $5^x + 11^y = z^2$. He concluded that the first equation has no solution while the second equation has no valid solutions when y is even. For all values $1 \leq x \leq 14$ and all odd values $1 \leq y \leq 9$, he identified exactly three solutions up to $5^{14} + 11^9 = 8461463316$. Further, Burshtein (2020) extended his inquiry into the equation $p^x + q^y = z^4$ and established that there are infinitely many solutions when $p = 2$ with equal values of x and y. However, he found that if x and y are distinct, there are no solutions, nor for any primes $p > 2$.

Sapar and Yow (2021) identified general forms of non-negative integral solutions to the equation $x^2 + 8(7^b) = y^{2r}$ under various conditions. Dokchan and Pakapongpun (2021) demonstrated that the Diophantine equation $p^x + (p+20)^y = z^2$ has no positive integers solutions for x,y and z with both p and $(p+20)$ are prime numbers. Borah and Dutta (2022) published findings on the exponential Diophantine equation $7^x + 32^y = z^2$, where they found discovered a unique solution $(x, y, z) = (2,1,9)$ Additionally, for the Diophantine equation $2^x + 7^y = z^2$ when $x \neq 1$, they found two solutions, which are (3,0,3) and (5,2,9). Yow et. al. (2022) provided bounds on the number of non-negative integral solutions for each b related to the Diophantine equation $x^2 + 16(7^b) = y^{2r}$. Burshtein (2017) suggested exploring solutions for the Diophantine equation $p^x + (p+4)^y = z^2$ under the condition $x + y > 4$ in future research. Consequently, this paper aims to find integral solutions to the equation $p^x + q^y = z^2$ with the condition $x + y = 5$.

RESULTS

In this section, we will seek an integral solution to the Diophantine equation $p^x + q^y = z^2$ with (p, q) being twin primes and $x + y = 5$. The result is shown below:

Theorem 1: Suppose x and y are positive integers, while p and q are odd primes. There are no non-trivial solutions to the Diophantine equation $p^x + q^y = z^2$ when (p, q) are twin primes and $x + y = 5$.

Proof:
Assume p is an odd prime and (p, q) are twin primes, where $q = p + 2$ and $p < q$. Refereeing to the equation

$$p^x + q^y = z^2 \qquad (1)$$

we will examine four cases such that the sum of x and y equal to 5, as shown in Table 1.

Table 1: Possible combination of x and y for equation (1)

Case	$\boldsymbol{x}$	**y**	$\boldsymbol{x+y}$
I	1	4	5
II	2	3	5
III	3	2	5
IV	4	1	5

Case I: Let $x = 1$ and $y = 4$. Substituting these values into equation (1) yields:

$$p + q^4 = z^2 \qquad (2)$$

Next, let $m = q^2$. Since p and q are twin primes, with $p < q$, it follows that $p < m$. Substituting m into equation (2) gives us $p = (z + m)(z - m)$. Assuming the left-hand side (LHS) equals the righ-hand side (RHS), we set $p = z + m$ and $1 = z - m$. By solving these equations simultaneously, we find $z = \frac{p+1}{2}$. Substituting this back into equation (2), and considering that q is a twin prime, we simplify to obtain

$$4p^4 + 32p^3 + 95p^2 + 130p + 63 = 0. \qquad (3)$$

Upon solving equation (3), we conclude that there are integral solutions since p must be an odd prime such that (p,q) are twin primes. Specifically, we find two real solutions, $p \approx -0.415$ and -31.781, as well as two complex solutions. Consequently, there is no non-trivial integral solution to the equation $p^x + q^y = z^2$ for $x = 1$ and $y = 4$ when (p, q) are twin primes.

Case II: Let $x = 2$ and $y = 3$. Substituting these values into equation (1) results in

$$p^2 + q^3 = z^2. \qquad (4)$$

By substituting $q = p + 2$ into (4), we derive

$$p^3 + 7p^2 + 12p + 8 = z^2. \qquad (5)$$

Factoring equation (5), leads to

$$(p^2 + 12)(p + 7) = \left(z - \sqrt{76}\,\right)\left(z + \sqrt{76}\,\right).$$

Now, let's define $p^2 + 12 = z - \sqrt{76}$ and $p + 7 = z + \sqrt{76}$. by solving these two equations simultaneously, we orrive at;

$$p^2 - p + 5 + 2\sqrt{76} = 0.$$

Clearly, p is not an integer. Thus, there is no non-trivial solution to the equation $p^x + q^y = z^2$ for $x = 2$ and $y = 3$.

Case III: Assume $x = 3$ and $y = 2$. By substituting these values into equation (1), we obtain

$$p^3 + q^2 = z^2. \quad (6)$$

Substituting $q = p + 2$ into equation (6) and simplifying it, we get

$$(p+1)(p^2+4) = z^2.$$

Since ringh-hand side (RHS) must be a perfect square, it is evident that there are no non-trivial solutions (p, q, z) to the equation $p^x + q^y = z^2$ for $(x, y) = (3, 2)$ and (p, q) are twin primes.

Case IV: Assume $x = 4$ and $y = 1$. Substituting these values into equation (1), we obtain

$$p^4 + q = z^2. \quad (7)$$

By substituting $q = p + 2$ into equation (7), we derive

$$p^4 + (p + 2) = z^2.$$

It is clear that the left-hand side cannot be expressed as a perfect square, as there is no perfect square between two consecutive squares. Therefore, there are no non-trivial solutions to the equation $p^x + q^y = z^2$ for $x = 4$ and $y = 1$.

The following theorem, will examine the equation $p^x + q^y = z^2$ when (p, q) are cousin primes and $x + y = 5$, with the result presented as follows:

Theorem 2: Let x and y be positive integers, and p and q be odd primes. There are no non-trivial solution to the equation $p^x + q^y = z^2$ when (p, q) are cousin primes and $x + y = 5$.

Proof:
Assume p is an odd prime, and let (p, q) be cousin primes such that $q = p + 4$. Based on the equation

$$p^x + q^y = z^2 \quad (8)$$

we will evaluate four cases where $x + y = 5$ as outlined in Table 1.

Case I: Let $x = 1$ and $y = 4$. Substituting x and y into equation (8) gives us:

$$p + q^4 = z^2. \tag{9}$$

Using a similar approach as in Case of Theorem 1, we conclude that there is no non-trivial solution to the Diophantine equation $p^x + q^y = z^2$ for $x = 1$ and $y = 4$.

Case II: Let $x = 2$ and $y = 3$. Substituting these values into equation (8) yields:

$$p^2 + q^3 = z^2.$$

Substituting $q = p + 4$ into equation (8) and simplifying gives:

$$p^2 + (p + 4)^3 = z^2.$$

By employing a similar method as in Case II of Theorem 1, we find there are no non-trivial solution to the equation $p^x + q^y = z^2$ for $x = 2$ and $y = 3$ when (p, q) are cousin primes.

Case III: Assume $x = 3$ and $y = 2$. Substituting these values into equation (8) and with $q = p + 4$, we derive:

$$p^3 + p^2 + 8p + 16 = z^2.$$

Using a similar argument as Case III of Theorem 1, we conclude that there is no non-trivial solution to the equation $p^x + q^y = z^2$ for $x = 3$ and $y = 2$ with (p, q) beingcousin primes.

Case IV: Assume $x = 4$ and $y = 1$. Substituting these values and $q = p + 4$ into equation (8) yields:

$$p^4 + (p + 4) = z^2.$$

Employing a similar method and arguments as in Case IV in Theorem 1, we confirm that there is no non-trivial solution to the equation $p^x + q^y = z^2$ for $x = 4$ and $y = 1$ when (p, q) are cousin primes.

Having examined all cases, we have demonstrated that no non-trivial solutions exist for the Diophantine equation $p^x + q^y = z^2$ when (p, q) are cousin primes and $x + y = 5$.

Next, we will seek a solution to the equation $p^x + q^y = z^2$ where (p, q) are sexy primes and $x + y = 5$. The result is provided in the following theorem.

Theorem 3: Let x and y be positive integers, and p and q be odd primes. There are no non-trivial solutions to the equation $p^x + q^y = z^2$ when (p, q) are sexy primes and $x + y = 5$.

Proof:
Assume p is an odd prime and let (p, q) be sexy primes such that $q = p + 6$ and $p < q$. We will examined four cases where the sum of x and y equals 5:

Case I: Let $x = 1$ and $y = 4$. Substituting these values into equation (1) gives us:

$$p + q^4 = z^2 \tag{10}$$

Using a similar approach as in Case I in Theorem 1, we arrive at the conclusion that there are no non-trivial solutions to the equation $p^x + q^y = z^2$ for $x = 1$ and $y = 4$. Thus, therefore is no non-trivial solution to $p^x + q^y = z^2$ for $(x,y)=(1, 4)$ when (p, q) are sexy primes.

Case II: Let $x = 2$ and $y = 3$. Substituting these values into equation (1) results in:

$$p^2 + q^3 = z^2.$$

Now, substituting $q = p + 6$ into equation (10) and simplifying, we obtain:

$$p^3 + 19p^2 + 108p + 216 = z^2.$$

Using a similar method as in Case II of Theorem 1, we find that there are no non-trivial integral solutions to the equation $p^x + q^y = z^2$ for $(x,y)=(2,3)$ when (p, q) are sexy primes.

Case III: Let $x = 3$ and $y = 2$. By substituting these values and $q = p + 6$ into equation (10), we get:

$$p^3 + p^2 + 12p + 36 = z^2.$$

As in Case III of Theorem 1, we can conclude that there is no non-trivial integral solutions to the equation $p^x + q^y = z^2$ for $(x,y)=(3,2)$ when (p, q) are sexy primes.

Case IV: Let $x = 4$ and $y = 1$. Substituting these values along with $q = p + 6$ into equation (10) gives us:

$$p^4 + (p + 6) = z^2.$$

By applying a similar method as in Case IV of Theorem 1, we can confirm that there is no non-trivial solutions to the Diophantine equation $p^x + q^y = z^2$ for $(x,y)=(4,1)$ when (p, q) are sexy primes.

After evaluating all cases, we have demonstrated that there are no non-trivial solutions to the Diophantine equation $p^x + q^y = z^2$ when (p, q) are sexy primes and $x + y = 5$.

In the subsequent theorem, we will seek an integral solution to the Diophantine equation $a^x + b^y = z^2$ for any $a, b \in \mathbb{Z}^+$, where the sum of x and y is fixed. The possible combination are provided in Table 2.

Table 2: Possible combination of x and y

Case	$\boldsymbol{x}$	**Y**	$\boldsymbol{x+y}$
I	1	4	5
Ii	2	3	5

To begin, we examine Case I: Let $x = 1$ and $y = 4$.

Theorem 4: Let a, b, x, y and z be positive integers. The solutions to the equation $a^x + b^y = z^2$ for $(x, y) = (1,4)$ are given by $(a, b, z) = (2m^2 + 1, m, m^2 + 1)$, where m is a positive integer.

Proof:
Consider the equation

$$a^x + b^y = z^2 \qquad (11)$$

where a, b, x, y and z being positive integers. Setting $(x, y) = (1,4)$ and substituting into equation (11), we have

$$a + b^4 = z^2$$

Let $b = m$. Then we rewrite the equation as

$$a + m^4 = z^2$$

Now, we can express a as:

$$a = (z + m^2)(z - m^2).$$

Since right-hand side (RHS) equals the left-hand side (LHS), by comparing both sides, we derive the following equations:

$$a = z + m^2 \qquad (12)$$

$$1 = z - m^2. \qquad (13)$$

By solving equations (12) and (13) simultaneously, we obtain:

$a = 2m^2 + 1$ and $z = m^2 + 1$

Thus, we have

$(a, b, z) = (2m^2 + 1, m, m^2 + 1)$

where m is a positive integer.

Now, we will consider Case II, where $(x, y) = (2,3)$. To determine the solution, we will consider the parity of b as stated in the following theorem.

Theorem 5: Let a, b, x, y and z be positive integers. The solution to the equation $a^x + b^y = z^2$ with $(x, y) = (2,3)$ are given by:

$$(a, b, z) = \begin{cases} (2m^2 + m, 2m + 1, 2m^2 + 3m + 1) & \text{if } b \text{ odd} \\ (2m^2 - m, 2m, 2m^2 + m) & \text{if } b \text{ even.} \end{cases}$$

where m is a positive integer.

Proof:
Let $(x, y) = (2,3)$ and substitute into equation (11):

$a^2 + b^3 = z^2$.

Assuming b is an odd integer, we can set $b = 2m + 1$:

$(2m + 1)^3 = (z + a)(z - a)$.

Since right-hand side (RHS) equals left-hand side (LHS), we can let

$z + a = (2m + 1)^2$ and $z - a = (2m + 1)$.

Solving these equations simultaneously, we find:

$z = (m + 1)(2m + 1)$ and $a = m(2m + 1)$

This results in the same outcome if we reverse the assignments of $z - a$ and $z + a$. Therefore,

$(a, b, z) = (2m^2 + m, 2m + 1, 2m^2 + 3m + 1)$

where m is a positive integer.

Now, let us consider $(x, y) = (2,3)$ and substitute into equation (11), we get $a^2 + b^3 = z^2$ and the case where $b = 2m$ is an even integer. In this scenario, we have:

$$(2m)^3 = (z + a)(z - a)$$

Since the right-hand side (RHS) equals the left-hand side (LHS), we set:

$$z + a = (2m)^2 \text{ and } z - a = (2m)$$

Solving these equations simultaneously, we get:

$$z = m(2m + 1) \text{ and } a = m(2m - 1)$$

We will arrive at the same result if we switch the values for $z - a = (2m)^2$ and $z + a = (2m)$. Thus, the solution to the equation $a^x + b^y = z^2$ with $(x, y) = (2,3)$ is:

$$(a, b, z) = (m(2m - 1), 2m, m(2m + 1))$$

where m is positive integer.

CONCLUSION

From this study, we established that that there are no non-trivial integral solutions to the Diophantine equation $p^x + q^y = z^2$ when (p,q) are twin, cousin, or sexy primes and $x + y = 5$. For the Diophantine equation $a^x + b^y = z^2$ where $a, b \in \mathbb{Z}^+$ and $(x, y) = (2,3)$, the solutions are given by $(a, b, z) = (2m^2 + m, 2m + 1, 2m^2 + 3m + 1)$ for odd b odd, and $(a, b, z) = (2m^2 - m, 2m, 2m^2 + m)$ for even b. For the case when $(x, y) = (1,4)$, the solution is $(a, b, z) = (2m^2 + 1, m, m^2 + 1)$, where m is a positive integer. This study could be expanded to explore integral solutions for cases involving prime triplets, prime quadruplets, or other configurations of prime number.

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